Since $\partial/\partial x_{n+1} \notin D_r$,

$$\sum_{s=1}^{r-1}(-1)^{s+1}\left(a_{l_j-r-s+1,\,l_k-r+s+1} + a_{l_k-r+s+1,\,l_j-r-s+1}\right)$$

$$-\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k-r-s+2,\,l_j-r+s} + a_{l_j-r+s,\,l_k-r-s+2}\right)$$

$$= (-1)^r\left(c^k_{l_j-2r+1} + c^j_{l_k-2r+1}\right)$$

$$\forall j, k = 1, \ldots, m \qquad \forall r = 0, \ldots, k_m - 1$$

On the other hand,

$$\left[\eta^j_r, \eta^k_r\right] = \left[\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_j-r-s+1,\,l_k-r+s} + a_{l_k-r+s,\,l_j-r-s+1}\right)\right.$$

$$-\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k-r-s+1,\,l_j-r+s} + a_{l_j-r+s,\,l_k-r-s+1}\right)$$

$$\left.+ (-1)^r\left(c^k_{l_j-2r} - c_{l_k-2r}\right)\right]\frac{\partial}{\partial x_{n+1}}$$

This implies

$$\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_j-r-s+1,\,l_k-r+s} + a_{l_k-r+s,\,l_j-r-s+1}\right)$$

$$-\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k-r-s+1,\,l_j-r+s} + a_{l_j-r+s,\,l_k-r-s+1}\right)$$

$$= (-1)^{r-1}\left(c^k_{l_j-2r} - c_{l_k-2r}\right)$$

$$\forall j, k = 1, \ldots, m \qquad \forall r = 0, \ldots, k_m - 1 \quad \square$$

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Spin-Axis Stabilization of a Rigid Spacecraft Using Two Reaction Wheels

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Introduction

DESIGN of control laws for underactuated spacecraft is important in practice. When one or more actuators fail, the attitude stabilization should be performed by using control laws for underactuated spacecraft with the remaining actuators. The problem of stabilization of a rigid spacecraft using two control inputs has been addressed by several researchers.[1,2] Recently, a discontinuous feedback control law was provided for the stabilization of a spacecraft about any equilibrium attitude.[1] In Ref. 1, it was conjectured without formal proof that the closed-loop system of a spacecraft with the proposed control law might be globally and asymptotically stable. Tsiotras and Longuski presented a new methodology for constructing feedback control laws for the attitude stabilization about the symmetry axis.[2] However, most previous researchers have considered the problem of controlling a spacecraft using less than three thrusters.

On the other hand, it is well known that with less than three momentum wheels the system becomes uncontrollable.[3] When the total angular momentum vector of the spacecraft was assumed to be zero, Krishnan et al. considered the stabilization problem of an underactuated rigid spacecraft using two momentum wheels.[4] In this Note, we consider the problem of spin stabilization of a spacecraft about a specified inertial axis using two reaction wheels. We derive a feedback control law that globally and asymptotically stabilizes the spacecraft about a revolute motion along a specified inertial axis.

Problem Statement

Consider the rotational dynamics of a rigid spacecraft controlled by two reaction wheels. We assume that the body-fixed coordinates are selected to be coincident with the spacecraft principal axes and that the rotation axes of the reaction wheels are the spacecraft principal axes. The dynamics of the spacecraft are given by[5]

$$(I_1 - J_a)\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + h_2\omega_3 - u_1 \tag{1}$$

$$(I_2 - J_a)\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 - h_1\omega_3 - u_2 \tag{2}$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 - h_2\omega_1 + h_1\omega_2 \tag{3}$$

$$\dot{h}_1 = -J_a\dot{\omega}_1 + u_1 \tag{4}$$

$$\dot{h}_2 = -J_a\dot{\omega}_2 + u_2 \tag{5}$$

where I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft including the moment of inertial contributions of the reaction wheels, J_a is the moment of inertia of each reaction wheel about its spin axis, and ω_1, ω_2, and ω_3 are the components of the angular velocity vector along the principal axes of the body-fixed frame. Also, h_1 and h_2 are relative angular momenta of the reaction wheels with respect to their respective rotational axes, h_i are defined as J_a Ω_i, where Ω_i is the wheel speed with respect to the spacecraft, and u_1 and u_2 are the control torques of the reaction wheels. Without loss of generality, we assume that $I_i > J_a$.

In Ref. 6, Tsiotras and Longuski introduced a new parameterization defining the orientation of rotating rigid bodies. This parameterization is especially useful to the problem of the spin-axis stabilization. According to these results, a transformation from an inertial frame to a body-fixed frame can be achieved by first performing an

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initial rotation of magnitude z about, for example, the inertial three axis and then performing a second rotation to coincide the intermediate three axis to the body three axis. The second rotation can be characterized by w_1 and w_2.

In this Note, we consider the spin-axis stabilization of a spacecraft about a specified inertial axis using two reaction wheels. The final configuration corresponds to a uniform revolute motion about the inertial axis. In this case, the kinematic equations can be written as follows (omitting z):

$$\dot{w}_1 = w_2(\omega_2 w_1 + \omega_3) + \tfrac{1}{2}\omega_1\left(1 + w_1^2 - w_2^2\right) \tag{6}$$

$$\dot{w}_2 = w_1(\omega_1 w_2 - \omega_3) + \tfrac{1}{2}\omega_2\left(1 - w_1^2 + w_2^2\right) \tag{7}$$

Note that $w_1 = w_2 = 0$ indeed implies the stabilization of the spin axis, the three axis in this study, along an inertial reference.

Stabilizing Control Law

In this section, we present a feedback control law that stabilizes the partial states, ω_1, ω_2, w_1, and w_2. Such a control law can be used to stabilize a spacecraft about a revolute motion along a specified inertial axis.

Proposition 1: Consider the control law for the system [Eqs. (1–7)]:

$$u_1 = I_2\omega_2\omega_3 + h_2\omega_3 + k_1 w_1 + k_2\omega_1 \tag{8}$$

$$u_2 = -I_1\omega_1\omega_3 - h_1\omega_3 + k_1 w_2 + k_2\omega_2 \tag{9}$$

where k_1 and k_2 are constant control gains that satisfy $k_1 > 0$ and $k_2 > 0$. This control law ensures that the trajectories of the closed-loop system globally and asymptotically converge to the equilibrium point as follows:

$$\omega_1 = \omega_2 = w_1 = w_2 = 0 \tag{10}$$

Proof: Consider the subsystem of the four differential equations (1), (2), (6), and (7). Substitution of Eqs. (8) and (9) into Eqs. (1) and (2) yields

$$(I_1 - J_a)\dot{\omega}_1 = -I_3\omega_2\omega_3 - k_1 w_1 - k_2\omega_1 \tag{11}$$

$$(I_2 - J_a)\dot{\omega}_2 = I_3\omega_3\omega_1 - k_1 w_2 - k_2\omega_2 \tag{12}$$

Consider the following Lyapunov candidate function for the subsytem:

$$V = \tfrac{1}{2}(I_1 - J_a)\omega_1^2 + \tfrac{1}{2}(I_2 - J_a)\omega_2^2 + k_1 \ln\left(1 + w_1^2 + w_2^2\right) \tag{13}$$

The time derivative of V along the closed-loop trajectories is given by

$$\begin{aligned}\dot{V} &= (I_1 - J_a)\omega_1\dot{\omega}_1 + (I_2 - J_a)\omega_2\dot{\omega}_2 + \left[2k_1\Big/\left(1 + w_1^2 + w_2^2\right)\right] \\ &\quad \times (w_1\dot{w}_1 + w_2\dot{w}_2) = \omega_1(-I_3\omega_2\omega_3 - k_1 w_1 - k_2\omega_1) \\ &\quad + \omega_2(I_3\omega_3\omega_1 - k_1 w_2 - k_2\omega_2) + k_1(\omega_1 w_1 + \omega_2 w_2) \\ &= -k_2\left(\omega_1^2 + \omega_2^2\right) \le 0\end{aligned} \tag{14}$$

The Lyapunov candidate function fails to satisfy the asymptotic stability condition of Lyapunov's direct method because $\dot{V} = -k_2(\omega_1^2 + \omega_2^2)$ is only negative semidefinite. LaSalle's theorem can be used to establish the asymptotic stability of the subsystem.[7] To this end, we will first show that all trajectories remain bounded. From Eqs. (13) and (14), it is obvious that ω_1, ω_2, w_1, and w_2 are bounded. Let us show that ω_3, h_1, and h_2 do not blow up in finite time. In terms of basis vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, the total angular momentum vector of the spacecraft can be expressed as follows:

$$\boldsymbol{H} = (I_1\omega_1 + h_1)\hat{e}_1 + (I_2\omega_2 + h_2)\hat{e}_2 + I_3\omega_3\hat{e}_3 \tag{15}$$

Because no external torque is acting on the spacecraft, the total angular momentum vector of the spacecraft is conserved:

$$|\boldsymbol{H}|^2 = (I_1\omega_1 + h_1)^2 + (I_2\omega_2 + h_2)^2 + (I_3\omega_3)^2 = \text{const} \tag{16}$$

Because ω_1 and ω_2 are bounded, it is obvious from the preceding equation that the trajectories of h_1, h_2, and ω_3 are bounded. Therefore, we can conclude that all trajectories of the variables are bounded. Moreover, note that $\dot{V}$ is strictly negative everywhere except $\omega_1 = \omega_2 = 0$. If $\omega_1 = \omega_2 = 0$, then $\dot{\omega}_1 = \dot{\omega}_2 = 0$, and from Eqs. (11) and (12) $w_1 = w_2 = 0$. Therefore, the only trajectory that can stay identical at points where $\dot{V} = 0$ is the trajectory that $\omega_1 = \omega_2 = w_1 = w_2 = 0$. Therefore, from LaSalle's theorem the closed-loop trajectories are asymptotically stable about the equilibrium point in Eq. (10). In addition, the closed-system is globally and asymptotically stable because V is radially unbounded. □

Note that, if the total angular momentum of the spacecraft is not initially zero, then the angular momenta of two reaction wheels, h_1 and h_2, and ω_3 cannot converge to zero simultaneously according to the conservation of angular momentum. As $t \to \infty$, ω_3 will converge to a constant value ($\omega_{3\infty}$) from Eq. (3) and from Eqs. (4), (5), (8), and (9). Then the dynamics of the angular momenta of the two reaction wheels can be described by the following equations:

$$\dot{h}_1 = h_2\omega_{3\infty} \tag{17}$$

$$\dot{h}_2 = -h_1\omega_{3\infty} \tag{18}$$

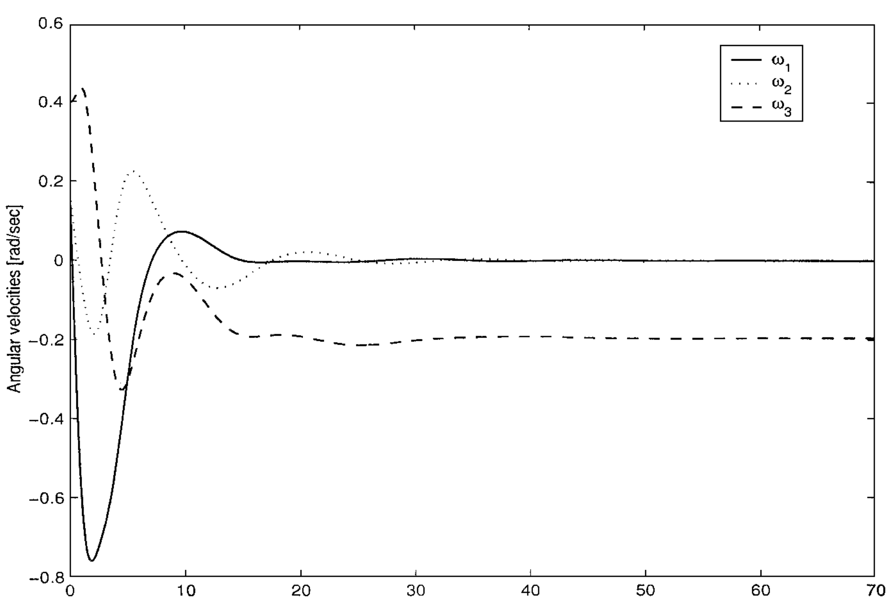

Fig. 1 Angular velocities.

By differentiating the preceding equations with respect to time, one can easily verify that h_1 and h_2 will oscillate periodically. Therefore, the control input, u_1 and u_2, will also oscillate from Eqs. (8) and (9).

Remark: As already noted, it is well known that a rigid spacecraft cannot be stabilized about an equilibrium point using two reaction wheels. In this Note, we presented a feedback control law that stabilizes the partial states of a spacecraft, that is, ω_1, ω_2, w_1, and w_2. This is the most one can expect in this case. If the total angular momentum of a spacecraft is initially nonzero, then there is no feedback control law that can make $(\omega_1, \omega_2, \omega_3, w_1, w_2) \to 0$. If the spacecraft is detumbled, then the angular momentum vectors of the two reaction wheels should constitute the total angular momentum vector. In this case, the spin axis should be always perpendicular to the total angular momentum vector. Therefore, we cannot detumble as well as stabilize the spin axis of the spacecraft about an arbitrary inertial axis using two reaction wheels.

Numerical Examples

Consider a spacecraft with the following moment of inertia (kilogram square meter):

$$(I_1, I_2, I_3) = (7, 10, 9)$$

The initial configuration of the spacecraft is assumed as follows:

$$(\omega_1, \omega_2, \omega_3) = (0.3, 0.2, 0.4), \qquad (w_1, w_2) = (2.0, 1.0) \quad (19)$$

where ω_i is in radians per second. We also assume that the two reaction wheels have an identical moment of inertia, $J_a = 0.0077\ \mathrm{kg \cdot m^2}$ and that the initial angular momenta of the wheels are zero. The control gains are selected as

$$k_1 = 3, \qquad k_2 = 3$$

Simulation results are shown in Figs. 1–3. Figure 1 shows that ω_1 and ω_2 converge to zero and that ω_3 converges to a constant value. The orientation of the spin axis is shown in Fig. 2. The spin axis has been successfully reoriented to a specified inertial axis. From Figs. 1 and 2, we can conclude that the spacecraft has been stabilized about a revolute motion along an inertial axis. The histories of the control inputs are shown in Fig. 3. As noted earlier, we can see that once the spacecraft was stabilized, the control commands oscillate periodically to conserve the total angular momentum.

Conclusions

It is well known that a rigid spacecraft cannot be stabilized about an equilibrium point using two reaction wheels. In this Note, we

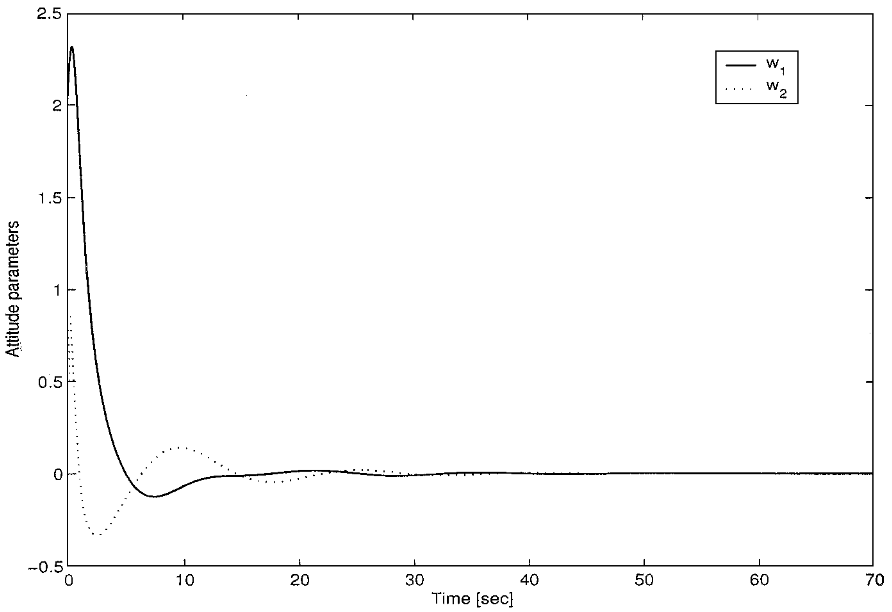

Fig. 2 Attitude parameters.

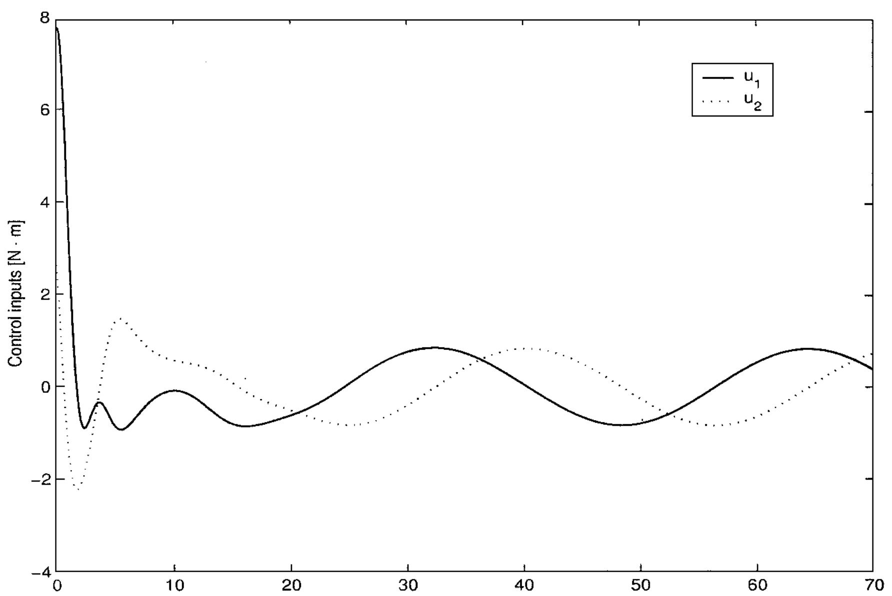

Fig. 3 Control inputs.

presented a feedback control law that globally and asymptotically stabilizes a spacecraft about a revolute motion along a specified inertial axis. After the spacecraft is stabilized, the command inputs of the feedback control law oscillate periodically to conserve the system angular momentum. In general there is no feedback control law that can detumble and reorient the spacecraft simultaneously.

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Uniform Damping of Plates Using a Modified Nodal Control Method

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I. Introduction

THE ultimate goal of many vibration control schemes is to dampen all of the active modes participating in the system response. In addition, one may want to preserve the uncontrolled mode shapes and frequencies. The main advantage in doing this lies in the energy savings that accompanies modal preservation. Finally, one may want an analytically sound method for actuator and sensor placement. Toward this end, the natural control theory provides a means for damping a structure, whereas preserving the modes and the node control theory provides a means for actuator and sensor placement.[1–3] Weaver and Silverberg applied this combination to actively damp a beam.[4] The strategy states that one can control the lowest N modes participating in a response by placing discrete sensor/actuator pairs at the nodes of the $(N+1)$ mode.[4] In doing this there is a natural decoupling that leads to a controlled system with the following properties: 1) frequency and modal invariance and 2) uniform damping.[4] Washington and Silverberg[5] added bias or steady-state calibration to this work, and Rosetti and Sun[6] extended the work to rings. Although the node control theory has been used extensively in beams and rings, a general proof of the theory was not discussed until recently.[7,8] Now that there is a mathematical basis for this theorem, its application to plates, shells, and general problems can now be investigated. The work in this Note applies the theory to rectangular plates. There are a myriad of uses associated with uniform damping nodal control (UDNC) as it applies to aircraft structures because the boundary condition of dominant local vibration of the aircraft can be modeled as combination of fixed and simply supported (SS) boundary conditions. Two important contributions are actuator placement for active noise cancellation in aircraft cabin enclosures and actuator placement for vibration control of next generation flexible aircraft structures.

In its application to plates, the work in this study postulates that the lowest M and N modes of vibration are controlled by $M \times N$ pairs of discrete actuators/sensors located at the intersections of the nodal lines of the $(M+1)\times(N+1)$ mode. The overall benefits of the method are as follows: 1) discrete sensors and actuators are used instead of distributed modal sensors and actuators.[9–11] 2) The method is resistant to control spillover[4,5] because the actuator/sensor pairs are located at the nodes of the next highest mode, which diagonalizes the control matrices. 3) Control moves are not calculated in modal domain but in the physical domain. 4) All modes are designed to be damped at the same rate thereby conserving energy. 5) The closed-loop frequencies are equivalent to open-loop frequencies. 6) The original mode shapes of the system are preserved. This work begins with the application of UDNC to SS plates. The methodology is then applied to the clamped-simply supported-clamped-simply supported (C-SS-C-SS) case. The Ritz method is then used to handle cases where an analytical solution does not exist. The paper then concludes with discussion.

II. Uniform Damping Control Method

A. Uniform Damping Control of Plates with Distributed Actuators/Sensors

The control term $f(x, y, z)$ is presented in the right-hand side of the following governing equation for rectangular plates:

$$\rho\frac{\mathrm{d}^2 w(x,y,t)}{\mathrm{d}t^2} + D\nabla^4 w(x,y,t) = f(x,y,t) \tag{1}$$

In Eq. (1) the displacement field and excitation terms can be written by the eigenfunction superposition method[12] as the following:

$$w(x,y,t) = \sum_m^\infty \sum_n^\infty W_{mn}(x,y)\eta_{mn}(t) \tag{2}$$

$$f(x,y,t) = \sum_m^\infty \sum_n^\infty W_{mn}(x,y)f_{mn}(t) \tag{3}$$

$$f_{mn}(t) = \iint_{x\ y} W_{mn}(x,y)f(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \tag{4}$$

where the subscripts m and n represent the mth mode in the x direction and nth mode in the y direction. The control force is expressed as

$$f(x,y,t) = gw(x,y,t) + h\dot{w}(x,y,t) \tag{5}$$

where g and h are the uniform damping coefficients.[12] The g and h coefficients are proportional to the measured displacement and velocity, respectively. Uniform damping control for each individual mode can be obtained by employing the principles of uniform damping[3] into Eq. (4):

$$f_{mn}(t) = \iint_{x,y} W_{mn}\,g\sum_r\sum_s W_{\mathrm{rs}}\eta_{\mathrm{rs}}(t)\,\mathrm{d}x\,\mathrm{d}y + \iint_{x,y} W_{mn}h\sum_r\sum_s W_{\mathrm{rs}}\dot{\eta}_{\mathrm{rs}}(t)\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

Substituting Eqs. (2–4) into Eq. (1), utilizing Eq. (6), and applying the two-dimensional orthonormality relationships yields the following equation:

$$\ddot{\eta}_{mn}(t) + \lambda_{mn}\eta_{mn}(t) = g\eta_{mn}(t) + h\dot{\eta}_{mn}(t) \tag{7}$$

If distributed sensors and actuators are used, the resulting gain matrix of the preceding equation can be shown to be diagonal. Here M and N are the largest number of modes considered to participate in the system in the x and y directions, respectively. When discrete actuators and sensors are used; however, the control gain matrices

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